

INTEGRAL OPERATORS IN BILATERAL GRAND LEBESGUE SPACES

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Abstract.

In this paper we estimate the norm of operator acting from one Bilateral Grand Lebesgue Space (BGLS) into other Bilateral Grand Lebesgue Space.

We also give some examples to show the sharpness of offered inequalities.

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1. INTRODUCTION

Let (S_1, Σ_1, μ_1) and (S_2, Σ_2, μ_2) be two measurable spaces with sigma-finite non-trivial measures μ_1, μ_2 . We denote as usually for any measurable function $f : \Sigma_1 \rightarrow R$

$$|f|_{p, \mu_1} = \left[\int_{S_1} |f(x)|^p d\mu_1(x) \right]^{1/p}, p \in [1, \infty),$$

$$f \in L_p(\mu_1) \Leftrightarrow |f|_{p, \mu_1} < \infty;$$

$$|f|_{p, \mu_2} = \left[\int_{S_2} |f(x)|^p d\mu_2(x) \right]^{1/p}, p \in [1, \infty),$$

or simple

$$|f|_{p, \mu} = \left[\int_S |f(x)|^p d\mu(x) \right]^{1/p}, p \in [1, \infty),$$

$f \in L_p \Leftrightarrow |f|_p < \infty$ in the case when $(S_1, \Sigma_1, \mu_1) = (S_2, \Sigma_2, \mu_2) = (S, \Sigma, \mu)$; and denote $L(a, b) = \cap_{p \in (a, b)} L_p$.

As usually,

$$|f|_{\infty, \mu} = \operatorname{vraisup}_{x \in X} |f(x)|.$$

We will denote for simplicity in the case when $X \subset R^d, d = 1, 2, \dots$ and when μ is usually Lebesgue measure

$$|f|_p = |f|_{p,\mu} = \left[\int_X |f(x)|^p dx \right]^{1/p}.$$

Let T be linear integral operator of a view:

$$T[f](s) = Tf(x) = \int_{S_2} K(s, s_2) f(s_2) d\mu(s_2), \quad s \in S_1. \quad (1)$$

Here the kernel $K = K(s_1, s_2)$, $s_1 \in S_1, s_2 \in S_2$ is bimeasurable, i.e. measurable relative the sigma-algebra $\Sigma_1 \times \Sigma_2$ with values in the real axis R function. We denote as $|K|_{p,q}$ the norm of the operator T from the space L_{p,μ_2} into the space L_{q,μ_1} :

$$|T|_{p,q} = \sup_{f \in L_{p,\mu_2}, f \neq 0} \frac{|T[f]|_{q,\mu_1}}{|f|_{p,\mu_2}}. \quad (2)$$

It is evident that in general case the function $K = |K|_{p,q}$ may be infinite for some values (p, q) , therefore we denote

$$G(T) = \{(p, q) : p, q \in [1, \infty), |T|_{p,q} < \infty\}$$

and suppose $G(T) \neq \emptyset$.

We define formally for the values (p, q) which does not belonging the set $G(T)$ $|T|_{p,q} = +\infty$.

Denote also for $j = 1, 2, 3$

$$G^{(j)}(K) = \{(p, q) : p, q \in [1, \infty), |K|_{p,q}^{(j)} < \infty\}$$

It is evident that $G(T) \supset G^{(j)}(K)$.

The complete description of the set $G(T)$ it follows from the classical interpolation Riesz-Thorin theorem, [1], [3].

There are many estimations of the value $T_{p,q}$; see, for example, [11], chapter 5,[24], chapter 7-9, [32], chapter 5. Let us denote for simplicity *the arbitrary upper estimation* of the value $|T|_{p,q}$ as $K_{p,q}^{(j)} : |T|_{p,q} \leq K_{p,q}^{(j)} = K_{p,q}$.

We recall here some expressions for the functionals $K_{p,q}^{(j)}$; obviously, these formulas are reasonable in the case of finiteness $K_{p,q}^{(j)}$.

For instance, $K_{p,p}^{(1)} =$

$$\left[\text{vraisup}_{s_2 \in S_2} \int_{S_1} |K(s_1, s_2)| d\mu(s_1) \right]^{1-1/p} \times \left[\text{vraisup}_{s_1 \in S_1} \int_{S_2} |K(s_1, s_2)| d\mu(s_2) \right]^{1/p}; \quad (3)$$

$$K_{q,r}^{(2)} = M_1^{1/p_1 - p_2/(pp_1)} \cdot M_2^{1/p}, \quad (4)$$

where $1 \leq p, q, p_1, q_1, p_2, q_2 < \infty$, $pp_1/(p - p_2) = r \in [1, \infty)$, $q \leq q_2$,

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1 = \frac{1}{p} + \frac{1}{q};$$

$$M_1 = \text{vraisup}_{s_1 \in S_1} \int_{S_2} |K(s_1, s_2)|^{p_2} d\mu_2(s_2),$$

$$M_2 = \text{vraisup}_{s_2 \in S_2} \int_{S_1} |K(s_1, s_2)|^{p_1} d\mu_1(s_1);$$

$$K_{p,r}^{(3)} = \left\{ \int_{S_1} d\mu_1(s_1) \cdot \left[\int_{S_2} |K(s_1, s_2)|^q d\mu_2(s_2) \right]^{r/q} \right\}^{1/r}, \quad (5)$$

$1/p + 1/q = 1$, $p, q, r \in (1, \infty)$ etc.

Evidently, we can define

$$K_{p,q}^{(4)} = \min_{j=1,2,3} K_{p,q}^{(j)}.$$

Another example (weight Hardy operator). Let $X = (a, b)$, $0 \leq a < b \leq \infty$ with ordinary Lebesgue measure $d\mu = dx$. Let us consider the following integral operator, which used in the theory of Sobolev spaces [30]:

$$T_a f(x) := v(x) \int_a^x u(t) f(t) dt.$$

Let $1 < q < p \leq \infty$; $p' = p/(p-1)$, $q' = q/(q-1)$, $1/s := 1/q - 1/p$, $B = B(p, q) :=$

$$\left\{ \int_a^b \left(\left(\int_x^b |v(t)|^q dt \right)^{1/q} \left(\int_a^x |u(t)|^{p'} dt \right)^{1/q'} \right)^s |u(x)|^{p'} dx \right\}^{1/s}.$$

It is known (see [12]) that

$$q^{1/q} (p' q/s)^{1/q'} B \leq |T_a|_{p,q} \leq q^{1/q} (p')^{1/q'} B.$$

In the case when $a = 0, b = \infty, u(t) = 1, v(x) = 1/x$ we obtain the classical Hardy operator

$$T_0[f](x) := x^{-1} \int_0^x f(t) dt$$

with the known exact value of $L_p \rightarrow L_p$ norm:

$$|T_0|_{p,p} = p/(p-1), \quad p \in (1, \infty],$$

see [19], p. 229-238. There are many analogous examples at the same place, e.g.:

$$T_+[f](x) := \int_0^\infty \frac{f(y) dy}{x+y}, \quad |T_+|_{p,p} = \frac{\pi}{\sin(\pi/p)}, \quad p \in (1, \infty);$$

$$T_m[f](x) := \int_0^\infty \frac{f(y) dy}{\max(x, y)}, \quad |T_m|_{p,p} = \frac{p^2}{p-1}, \quad p \in (1, \infty),$$

$$T_d[f](x) := \int_x^\infty \frac{f(y) dy}{y}, \quad |T_d|_{p,p} = p, \quad p \in (1, \infty),$$

$$T_l[f](x) = \int_0^\infty \frac{\log(x/y) f(y) dy}{x-y}, \quad |T_l|_{p,p} = \left[\frac{\pi}{\sin(\pi/p)} \right]^2,$$

etc.

Another examples. Let us consider the operators of a "multiplicative" view:

$$T_M[f](x) = T_M f(x) = \int_0^\infty K(x \cdot y) f(y) dy, \quad K(x) \geq 0.$$

Denote by

$$\zeta(s) = \int_0^\infty K(x) x^{s-1} dx$$

the Mellin's transform of the kernel $K(\cdot)$, and assume that there exist a values a, b ; $1 \leq a < b \leq \infty$ for which

$$\forall p \in (a, b) \Rightarrow \zeta(1/p) < \infty.$$

It is known [19], p. 256, [31], p. 146 that

$$|T_M[f]|_p^p \leq \zeta(1/p) \int_0^\infty (x|f(x)|)^p dx/x^2$$

and analogously

$$\int_0^\infty x^{p-2} |T_M[f](x)|^p dx \leq \zeta^p((p-1)/p) |f|_p^p.$$

In the case when $K(x) = \exp(-x)$ we obtain the classical Laplace transform:

$$\Lambda[f](x) = \Lambda f(x) = \int_0^\infty \exp(-xy) f(y) dy.$$

In this case $\zeta(s) = \Gamma(s)$ and moreover

$$|\Lambda[f]|_{p'} \leq (2\pi/p')^{1/p'} |f|_p, \quad p \in [1, 2].$$

Analogous estimations are true for Fourier transform.

In the books [19], chapter 6 and [31], chapter 5 there are many other examples of integral operators with calculated or estimated norms in the classical Lebesgue spaces L_p .

There exists a possibility when for any value of variable p , $p \in (A, B)$ there exists (in general case) a *unique* value q , $q \in (1, \infty)$ for which $|T|_{p,q} < \infty$; see example further. We will denote in this case $T = T(\cdot) \in UC$ and will denote the correspondent function by $w : w = w(p) : W(p) := |T|_{p,w(p)} < \infty$ and

$$\forall t \neq w(p) \Rightarrow |T|_{p,t} = \infty.$$

Our aim is a generalization of the estimation (3), (4), (5) on the so - called Bilateral Grand Lebesgue Spaces $BGL = BGL(\psi) = G(\psi)$, i.e. when $f(\cdot) \in G(\psi)$ and to show the precision of obtained estimations by means of the constructions of suitable examples.

We recall briefly the definition and needed properties of these spaces. More details see in the works [15], [16], [20], [21], [33], [34], [26], [23], [25] etc. More about rearrangement invariant spaces see in the monographs [3], [27].

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a+0)$ and $\psi(b-0)$, with conditions $\inf_{p \in (a,b)} \psi > 0$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b)$.

The Bilateral Grand Lebesgue Space (in notation BGLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : R^d \rightarrow R$ endowed with the norm

$$\|f\|_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a,b)} \left[\frac{|f|_p}{\psi(p)} \right], \quad (6)$$

if it is finite.

In the article [34] there are many examples of these spaces. For instance, in the case when $1 \leq a < b < \infty, \beta, \gamma \geq 0$ and

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta} (b - p)^{-\gamma};$$

we will denote the correspondent $G(\psi)$ space by $G(a, b; \beta, \gamma)$; it is not trivial, non - reflexive, non - separable etc. In the case $b = \infty$ we need to take $\gamma < 0$ and define

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p - a)^{-\beta}, p \in (a, h);$$

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = p^{-\gamma} = p^{-|\gamma|}, p \geq h,$$

where the value h is the unique solution of a continuity equation

$$(h - a)^{-\beta} = h^{-\gamma}$$

in the set $h \in (a, \infty)$.

The $G(\psi)$ spaces over some measurable space (X, F, μ) with condition $\mu(X) = 1$ (probabilistic case) appeared in [26].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [9], [23].

It was proved also that in this case each $G(\psi)$ space coincides with the so - called *exponential Orlicz space*, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and *singular* integral operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

Remark 1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, p = r; \psi_r(p) = \infty, p \neq r, p, r \in (a, b)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(a, b) : 1 \leq a < b \leq \infty, \forall p \in (a, b) |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T$, T is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(a, b) : 1 \leq a < b \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [33].

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in the theory of Partial Differential Equations [15], [20], theory of probability in Banach spaces [29], [26], [33], in the modern non-parametrical statistics, for example, in the so-called regression problem [33].

The article is organized as follows. In the second section we obtain the main result: upper bounds for *integral* operators in the Bilateral Grand Lebesgue spaces. In the next sections we investigate some one-dimensional classical *singular* operators in BGLS: we obtain the upper estimations for its norm and consider some examples in order to show the sharpness of upper estimations.

Fourth section contains the asymptotically exact BGLS norm calculation for the integral operators with homogeneous kernel. In the sixth sections we investigate some integral operators over the spaces with weight.

The 7th section contains the investigation of Fourier integral operators in BGLS spaces with exact embedding constant calculations. In the eight section we obtain the exact norm estimation for the celebrated Riesz operator.

In the 9th section we investigate the case when the measures $\mu_{1,2}$ are pure atomic with unit measure of all points ("discrete case").

The last section contains some concluding remarks and generalizations.

We use symbols $C(X, Y)$, $C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \rightarrow R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol \sim will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, \quad I(x \in A) = 0, x \notin A;$$

here A is a measurable set.

All the passing to the limit in this article may be proved by means of Lebesgue dominated convergence theorem.

2. MAIN RESULT: NORM ESTIMATIONS FOR REGULAR INTEGRAL OPERATORS.

Let $\psi(\cdot) \in \Psi(a, b)$, where $1 \leq a < b$. Let T be the operator described in the first section. We introduce the function

$$\nu(p) = \inf_{q \in (a, b)} \{|T|_{p, q} \cdot \psi(q)\}. \quad (7)$$

Then the function $\nu(\cdot)$ belong to the space $G(\psi; c, d)$, where

$$c = \inf\{p, p \geq 1, \nu(p) < \infty\}; \quad d = \sup\{p, p \geq 1, \nu(p) < \infty\}.$$

For the function $\nu(\cdot)$ is true the elementary estimation

$$\nu(p) \leq \min_j \inf_{q \in (a,b)} \{ |K|_{p,q}^{(j)} \cdot \psi(q) \}. \quad (8)$$

Theorem 1. Let $f \in G(\psi)$, $\psi \in \Psi(a, b)$ and let T be the operator described in the first section. Then

$$\|T f\|_{G(\nu)} \leq \|f\|_{G(\psi)}. \quad (9).$$

On the other words, the operator T is bounded from the space $G(\psi)$ into the *another* space $G(\nu)$.

Proof of the theorem 1 is very simple. Denote for the simplicity $u = Tf$. We suppose $f(\cdot) \in G(\psi)$; otherwise is nothing to prove.

We can assume without loss of generality that $\|f\|_{G(\psi)} = 1$; this means that

$$\forall q \in (a, b) \Rightarrow |f|_q \leq \psi(q).$$

Using the definition of the norm $|T|_{p,q}$ we obtain:

$$|u|_p \leq |T|_{p,q} \cdot \psi(q), \quad q \in (a, b).$$

The assertion of theorem 1 follows after the dividing over the $\nu(q)$, taking the minimum over q , $q \in (a, b)$ and on the basis of the definition of the $G(\psi)$ spaces. \square

Corollary 1. We consider separately the case when $T \in UC$; then the assertion (9) of theorem 1 may be rewritten as follows:

$$|T f|_{w(p)} \leq \nu(p) \cdot |f|_p. \quad (10)$$

We investigate in the remainder part of this section the *sufficient* conditions for *compactness* of the operator T from the one Bilateral Grand Lebesgue Space to the other Bilateral Grand Lebesgue Space.

We assume here that both the measures μ_1, Σ_1 and μ_2, Σ_2 are separable relative the distances correspondingly

$$\rho_1(A_1, A_2) = \mu_1(A_1 \setminus A_2) + \mu_1(A_2 \setminus A_1), A_1, A_2 \in \Sigma_1;$$

$$\rho_2(B_1, B_2) = \mu_2(B_1 \setminus B_2) + \mu_2(B_2 \setminus B_1), B_1, B_2 \in \Sigma_2.$$

It follows from the famous Kondrashov theorem [24], chapter 9, section 3 that if $K_{p,q}^{(3)} < \infty$, then the operator T is compact as the linear operator from the space L_{q,μ_2} into the space L_{p,μ_1} .

For example, it is true for the so-called embedding Sobolev's operator.

But in the case of BGLS spaces still the embedding operator is bounded and is not compact [34].

Theorem 2. Let $\psi(\cdot) \in G(\psi; a, b)$, and let

$$\max(\lim_{p \rightarrow c+0} \nu(p), \lim_{p \rightarrow d-0} \nu(p)) = \infty. \quad (11)$$

Let $\zeta(\cdot)$ be arbitrary function from the space $G(\psi; c, d)$ such that $\zeta(\cdot) \gg \nu(\cdot)$. Then the operator T is compact operator from the space $G(\psi)$ into the space $G(\zeta)$.

Proof. It follows from the Kondrashov's theorem that the operator T is compact operator from the arbitrary space L_{p, μ_2} , $p \in (a, b)$ into the arbitrary space L_{q, μ_1} , $q \in (c, d)$. The proposition of theorem 2 follows from the main result of an article [34]. \square

3. CLASSICAL ONE-DIMENSIONAL SINGULAR OPERATORS

We consider in this section some classical one-dimensional integral operators with homogeneous kernel of degree -1 of Hardy-Littlewood-Young type. We intend to obtain the *exact* values of embedding constants or as a minimum to obtain the low bounds for the norms of the operator T in the BGLS spaces.

In detail, we consider the one-dimensional singular integral operators in the space $X = (0, \infty)$ of a view

$$T_H f(x) = \int_0^\infty K(x, y) f(y) dy, \quad (12)$$

where the kernel $K(\cdot, \cdot)$ is presumed to be non-negative and homogeneous of degree -1 ; this means that

$$K(x, y) = x^{-1} H(y/x), \quad x, y > 0,$$

where $H(\cdot)$ is some measurable function.

We define for the values $p \in (1, \infty)$ the function

$$\phi(p) = \phi_H(p) \stackrel{\text{def}}{=} \int_0^\infty z^{-1/p} H(z) dz \quad (13)$$

and suppose in this subsection the *finiteness* of the function $\phi(p)$ for all the values $sp \in (1, \infty)$.

The famous result belonging to Hardy and Littlewood [19], p. 93-114 states that

$$|T_H|_{p,p} \leq \phi(p).$$

and the last inequality is exact under some additional conditions. We can therefore use theorem 1. Namely, we define the arbitrary function $\psi(\cdot)$ from the set $G\Psi(1, \infty)$ the new function

$$\psi_{(\phi)}(p) = \phi(p)\psi(p).$$

It follows from theorem 1 that

$$\|T_H f\|_{G(\psi_{(\phi)})} \leq V_H \|f\|_{G(\psi)}, \quad V_H \leq 1. \quad (14)$$

We will prove that under some additional conditions the exact value for the constant V_H in the inequality (14) is equal to one.

Before the formulating the main result of this section, we need to establish some preliminary lemmas.

Lemma 1a. Assume that the (measurable non-negative) function $H(\cdot)$ satisfies the following conditions:

$$\exists \delta_0 > 0, \int_1^\infty z^{-1+\delta_0} H(z) dz < \infty; \quad (15a)$$

$$\begin{aligned} \exists H_- \in (0, \infty), z \rightarrow 0+ \Rightarrow H(z) = \\ H_- |\log z|^{\beta_1} S_1(|\log z|)(1 + 0(|\log z|^{-\gamma})), \end{aligned} \quad (15b)$$

where $H_- = \text{const} \in (0, \infty)$, $\beta_1 = \text{const} \geq 0$; $S_1(z)$ is non-negative continuous slowly varying as $z \rightarrow \infty$ functions:

$$\forall l > 0 \Rightarrow \lim_{z \rightarrow \infty} S_1(lz)/S_1(z) = 1.$$

We propose that as $p \rightarrow 1 + 0$

$$\phi_H(p) \sim H_-(p-1)^{-\beta_1+1} S_1(1/(p-1)) \Gamma(\beta_1 + 1). \quad (16)$$

Lemma 1b. Assume that the (measurable non-negative) function $H(\cdot)$ satisfies the following conditions:

$$\exists \delta_0 > 0, \int_0^1 z^{-\delta_0} H(z) dz < \infty; \quad (17a)$$

$$\begin{aligned} \exists H_+ \in (0, \infty), z \rightarrow 0+ \Rightarrow H(z) = \\ H_+ z^{-1} |\log z|^{\beta_2} S_2(|\log z|)(1 + 0(|\log z|^{-\gamma})), \end{aligned} \quad (17b)$$

where $H_+ = \text{const} \in (0, \infty)$, $\beta_2 = \text{const} \geq 0$; $S_2(z)$ is non-negative continuous slowly varying as $z \rightarrow \infty$ functions:

$$\forall l > 0 \Rightarrow \lim_{z \rightarrow \infty} S_2(lz)/S_2(z) = 1.$$

We propose that as $p \rightarrow \infty$

$$\phi_H(p) \sim H_+ p^{\beta_2+1} S_2(p) \Gamma(\beta_2 + 1). \quad (18)$$

Proofs. Proof of lemma 1a. We write first of all the partition

$$\phi_H(p) = \int_0^1 z^{-1/p} H(z) dz + \int_1^\infty z^{-1/p} H(z) dz = I_1 + I_2,$$

and note that the second integral is uniformly bounded for the values p nearest to $1 + 0$ by virtue of the condition (15a).

The assertion of lemma 1a it follows from the asymptotical equality: as $p \rightarrow 1 + 0$

$$\begin{aligned} \int_0^1 z^{-1/p} |\log z|^{\beta_1} S_1(\log z) dz \sim \\ (p-1)^{-\beta_1-1} S\left(\frac{1}{p-1}\right) \Gamma(\beta_1 + 1), \beta_1 \geq 0. \end{aligned}$$

The proof of the lemma 1.b may be obtained analogously.

Theorem 3. Suppose the function $H(\cdot)$ satisfies either the conditions of lemma 1a or the conditions of lemma 1b. Then

$$\|T_H f\| G(\psi(\phi)) \leq V_H \|f\| G(\psi), \quad (19)$$

where the exact value of constant V_H is equal to one.

Proof of the theorem 3.

A. Let us consider and investigate the following value:

$$\begin{aligned}
Z &\stackrel{def}{=} \sup_{\psi \in G\Psi(1, \infty)} \|T_H\| (G(\psi) \rightarrow G(\psi_{(\phi)})) = \\
&\sup_{\psi \in G\Psi(1, \infty)} \sup_{f \in G\psi} \frac{\|T_H[f]\| G(\psi_{(\phi)})}{\|f\| G(\psi)} = \\
&\sup_{\psi \in G\Psi(1, \infty)} \sup_{f \in G\psi} \frac{\sup_p [|T_H[f]|_p / (\psi(p)\phi(p))]}{\sup_p [|f|_p / \psi(p)]}.
\end{aligned} \tag{19}$$

Note that if we choose the function $\psi(\cdot)$ as follows:

$$\psi(p) = |f|_p,$$

i.e. the *natural* choice of a function $\psi(\cdot)$, obviously if the function f belong to the set $L(1, \infty)$, we obtain the following *low estimation* for the value Z :

$$Z \geq \sup_{f \in L(1, \infty)} \sup_{p \in (1, \infty)} \frac{|T_H f|_p}{\phi(p) |f|_p}. \tag{20}$$

B. It remains only to prove the low bound for the value V_H . Let us consider at first the case when the function $H(\cdot)$ satisfies the conditions of lemma 1a.

Let us choose the function, more exactly, the family of a functions

$$g_\Delta(x) = |\log x|^\Delta I(x \in (0, 1)),$$

here $\Delta = \text{const} \geq 2$ is some fixed number.

We have:

$$|g_\Delta|_p = \sqrt[p]{\Gamma(\Delta p + 1)}.$$

Further, we denote $v_\Delta = T_H g_\Delta$. We obtain as $x \rightarrow 0$:

$$\begin{aligned}
v_\Delta(x) &= \int_0^1 x^{-1} |\log y|^\Delta H(y/x) dy \sim \\
&\frac{H_+}{\Delta + \beta_2 + 1} |\log x|^{\beta_2 + \Delta + 1} S_2(|\log x|).
\end{aligned}$$

We obtain after some calculation

$$\overline{\lim}_{p \rightarrow \infty} \frac{|v_\Delta|_p}{\phi(p) |g_\Delta|_p} \geq \exp(-\beta_2 - 1) \left(1 + \frac{\beta_2 + 1}{\Delta}\right)^\Delta.$$

Therefore,

$$Z \geq \lim_{\Delta \rightarrow \infty} \exp(-\beta_2 - 1) \left(1 + \frac{\beta_2 + 1}{\Delta}\right)^\Delta = 1.$$

This completes the proof of theorem 3 in the case when the function $H(\cdot)$ satisfies the conditions of lemma 1b.

C. The second case. i.e. when the function $H(\cdot)$ satisfies the conditions of lemma 1a, the proof provided analogously by the consideration the family of a functions

$$f_\Delta(x) = x^{-1} (\log x)^\Delta I(x > 1).$$

We consider further in this section the integral operators with homogeneous kernel of degree -1 which does not satisfy the conditions of theorem 3. For instance, let the operator $T_{r,s}[f]$ has a view:

$$T_{r,s}[f](x) = \int_0^x \frac{(x-y)^{r-1} f(y) dy}{x^s y^{r-s}},$$

where $r, s = \text{const}$, $0 < r < s + 1$. Here

$$H(z) = z^{-(r-s)} (1-z)^{r-1} I(z \in (0, 1))$$

and correspondingly *only for the values* $p > 1/(1 + s - r)$

$$\phi_{r,s}(p) = B(1 - 1/p - (r - s), r) = \frac{\Gamma(1 - 1/p - (r - s)) \Gamma(r)}{\Gamma(1 - 1/p + s)},$$

where $B(\cdot, \cdot)$ denotes usually Beta-function.

Recall that we consider here the case when $p > 1$; so we assume that

$$p > \max(1, 1/(1 + s - r)),$$

in the contradiction to the theorem 3.

The family of the operators $\{T_{r,s}[f](x)\}$ contains the classical Rieman's fractional integral operator, up to multiplicative constant and the power factor x^t , which does not dependent on the variable y :

$$R_{(r)}[f](x) \frac{1}{\Gamma(r)} \int_0^x (y-x)^{r-1} f(y) dy.$$

Analogously may be considered the fractional integral in the Weil's sense:

$$W^{(r)}[f](x) \frac{1}{\Gamma(r)} \int_x^\infty (y-x)^{r-1} f(y) dy.$$

So, we will consider the operators $T_H[f]$ with homogeneous kernel

$$K(x, y) = x^{-1} H(y/x), \text{ degree}(K) = -1$$

where the function $H = H(z)$ does not satisfy the conditions of lemmas 1a or lemma 1b. Indeed, we consider as a functions $H(\cdot)$ the functions of a view

$$\mathbf{A} \ H(z) \sim H_- \cdot z^{\alpha-1} |\log z|^{\beta_1} S_1(z), \ z \rightarrow 0+$$

or

$$\mathbf{B} \ H(z) \sim H_+ \cdot z^{\alpha-1} |\log z|^{\beta_1} S_2(z), \ z \rightarrow \infty.$$

In both the cases $\alpha = \text{const} \in (0, 1)$ and as before $S_1(\cdot), S_2(\cdot)$ are non-negative continuous slowly varying as $z \rightarrow \infty$ functions.

Note that in considered cases **A** and **B** the function $\phi(p)$ does not exists in the whole axis $(1, \infty)$, in contradiction to the cases considered in the lemmas 1a and 1b. Namely, in the first case **A**

$$\phi_H(p) < \infty \leftrightarrow p > 1/\alpha.$$

and in the case of the condition **B**

$$\phi_H(p) < \infty \leftrightarrow p \in (1, 1/\alpha).$$

Lemma 2a.

Assume that the (measurable non-negative) function $H(\cdot)$ satisfies the following conditions:

$$\exists \delta_0 > 0, \int_1^\infty z^{-1+\delta_0} H(z) dz < \infty;$$

$$\exists H_- \in (0, \infty), z \rightarrow 0+ \Rightarrow H(z) =$$

$$H_- z^{\alpha-1} |\log z|^{\beta_1} S_1(|\log z|)(1 + 0(|\log z|^{-\gamma})),$$

where $H_- = \text{const} \in (0, \infty)$, $\beta_1 = \text{const} \geq 0$; $S_1(z)$ is non-negative continuous slowly varying as $z \rightarrow \infty$ functions.

We propose that as $p \rightarrow 1/\alpha + 0$

$$\phi_H(p) \sim H_- (p - 1/\alpha)^{-\beta_1+1} S_1(1/(p - 1/\alpha)) \Gamma(\beta_1 + 1).$$

Lemma 2b. Assume that the (measurable non-negative) function $H(\cdot)$ satisfies the following conditions:

$$\exists \delta_0 > 0, \int_0^1 z^{-\delta_0} H(z) dz < \infty;$$

$$\exists H_+ \in (0, \infty), z \rightarrow 0+ \Rightarrow H(z) =$$

$$H_+ z^{-1-\alpha} |\log z|^{\beta_2} S_2(|\log z|)(1 + 0(|\log z|^{-\gamma})),$$

where $H_+ = \text{const} \in (0, \infty)$, $\beta_2 = \text{const} \geq 0$; $S_2(z)$ is non-negative continuous slowly varying as $z \rightarrow \infty$ function:

We propose that as $p \rightarrow 1/\alpha - 0$

$$\phi_H(p) \sim H_+ (1/\alpha - p)^{-\beta_2+1} S_2(1/\alpha - p) \Gamma(\beta_2 + 1).$$

Proofs are analogous to the lemma 1a and may be omitted.

Theorem 4. Suppose the function $H(\cdot)$ satisfies either the conditions of lemma 2a or the conditions of lemma 2b. Then

$$||T_H f|| G(\psi(\phi)) \leq V_H ||f|| G(\psi), \quad (20)$$

where the exact value of constant V_H is equal to one.

Proof of the theorem 4 is at the same as the proof of theorem 3.

□

4. SINGULAR OPERATORS WITH ARBITRARY DEGREE OF KERNEL, WITH GENERALIZATION

A. We consider in this subsection the *family* of operators of a view

$$T_\lambda f(x) = \int_0^\infty \frac{f(y) dy}{(x+y)^\lambda}.$$

Here $\lambda = \text{const} \in (0, 1)$, $X = (0, \infty)$ and we denote $u(x) = T_\lambda f(x)$.

It is known [31], p. 215 that

$$|T_\lambda f|_{p/\lambda} \leq \left[\frac{\pi}{\sin(\pi/p)} \right]^\lambda |f|_{q/(q-\lambda)},$$

where $p \in (\lambda, \infty)$ and as ordinary $q = p/(p-1)$.

The last relation may be rewritten as follows.

$$|T_\lambda f|_p \leq \left[\frac{\pi}{\sin(\pi/(p\lambda))} \right]^\lambda \cdot |f|_{p/(p(1-\lambda)+1)},$$

but for the values p from the interval $p \in (1/\lambda, \infty)$.

In order to formulate the main result of this section, we need to introduce some functions. Let $h = h(\lambda)$, $\lambda \in (0, 1)$ be well-known entropy function

$$h(\lambda) = -\lambda \log(\lambda) - (1-\lambda) \log(1-\lambda).$$

Note that the function $h(\cdot)$ may be continued on the whole closed interval $[0, 1]$ as the continuous function, as long as

$$\lim_{\lambda \rightarrow 0+} h(\lambda) = \lim_{\lambda \rightarrow 1-0} h(\lambda) = 0.$$

Further, we define the function:

$$L(\lambda) := \max \{ \lambda \Gamma^\lambda (1 + 1/\lambda), e^{h(\lambda)} \},$$

$$V(\lambda, p) = \sup_{f \in L(1/\lambda, \infty), f \neq 0} \left\{ \frac{|T_\lambda f|_{p/\lambda}}{|f|_{p/(p(1-\lambda)+\lambda)}} \cdot \left[\frac{\sin(\pi/p)}{\pi} \right]^\lambda \right\},$$

$p \in (1, \infty)$;

$$\overline{V}(\lambda) = \sup_{p \in (1, \infty)} V(\lambda, p).$$

Note that

$$W \stackrel{\text{def}}{=} \sup_{\lambda \in (0, 1)} \overline{V}(\lambda) = 1.$$

Let also $\psi(\cdot)$ be arbitrary function from the space $G\Psi(1, 1/(1-\lambda))$. We define for any value $\lambda \in (0, 1)$

$$\psi_{(\lambda)}(p) := \left[\frac{\pi}{\sin(\pi/(p\lambda))} \right]^\lambda \cdot \psi \left(\frac{p}{p(1-\lambda)+1} \right).$$

Theorem 5.

$$\|T^{(1)} f\|_{G(\psi^{(1)})} \leq V_1 \|f\|_{G(\psi)}, \quad (21)$$

where the exact value of constant $V_1 = V_1(\lambda)$ lies in the closed interval

$$V_1 \in [L(\lambda), 1].$$

Proof. The upper bound for V_1 it follows from the $L_p \rightarrow L_p$ estimations for considered operator on the basis of theorem 1. To obtain the low bound $L(\lambda)$, we construct two examples.

FIRST EXAMPLE. Let us consider the function

$$f(x) = x^{-1}I(x > 1),$$

then

$$|f|_Q = (Q - 1)^{-1/Q}, \quad Q > 1.$$

and as $x \rightarrow \infty$

$$\begin{aligned} u_\lambda(x) &:= \int_1^\infty \frac{dy}{y(x+y)^\lambda} = x^{-\lambda} \int_{1/x}^\infty \frac{dz}{z(1+z)^\lambda} \sim \\ &x^{-\lambda} \int_{1/x}^1 z^{-1} dz = x^{-\lambda} \log x; \end{aligned}$$

and we compute ad $p \rightarrow 1 + 0$

$$|u_\lambda|_p \sim \frac{\Gamma^{1/p}(p+1)}{(p\lambda - 1)^{1+1/p}}.$$

The first example give us the following low estimation for the value V_1 :

$$V_1 \geq \lambda \Gamma^\lambda (1 + 1/\lambda).$$

SECOND EXAMPLE. We consider here the function

$$g(x) = x^{-(1-\lambda)}I(x \in (0, 1)).$$

We get as $p \rightarrow \infty$:

$$\begin{aligned} |g|_Q &\sim \left(\frac{1-\lambda}{\lambda} \right)^{1-\lambda} p^{1-\lambda}; \\ v(x) &:= \int_0^1 \frac{y^{-(1-\lambda)} dy}{(x+y)^\lambda} \sim |\log x|, \\ |v|_p^p &\sim \int_0^1 |\log x|^p dx = \Gamma(p+1). \end{aligned}$$

The second example give us the next low estimation for the value V_1 :

$$V_1 \geq e^{h(\lambda)}.$$

Thus,

$$V_1 \geq \max \left(e^{h(\lambda)}, \Gamma^\lambda (1 + 1/\lambda) \right), \quad \lambda \in (0, 1). \quad (22)$$

□

B. We consider here the *family* of operators of a view

$$u(x) = I_{\alpha, \beta, \lambda}[f](x) = I[f](x) = |x|^{-\beta} \int_{R^d} \frac{f(y) |y|^{-\alpha} dy}{|x-y|^\lambda}; \quad (23)$$

here $\alpha, \beta \geq 0, \alpha + \lambda < d$.

We define the function $q = q(p)$ as follows:

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \gamma}{d}.$$

We will denote the set of all such a values (p, q) as $G(\alpha, \beta, \lambda)$ or for simplicity $G = G(\alpha, \beta, \lambda)$.

FURTHER WE WILL SUPPOSE IN THIS SUBSECTION THAT $(p, q) \in G(\alpha, \beta, \lambda) = G$.

We denote also

$$p_- := \frac{d}{d - \alpha}, \quad p_+ := \frac{d}{d - \alpha - \lambda};$$

and correspondingly

$$q_- := \frac{d}{\beta + \lambda}, \quad q_+ := \frac{d}{\beta},$$

where in the case $\beta = 0 \Rightarrow q_+ := +\infty$;

$$\kappa = \kappa(\alpha, \beta, \lambda) := (\alpha + \beta + \lambda)/d.$$

Let $\psi(\cdot) \in G\Psi(p_-, p_+)$; we define

$$\psi_{\alpha, \beta, \lambda} = [(p - p_-)(p_+ - p)]^{-\kappa} \psi(p). \quad (24)$$

Theorem 6. There exist a two positive finite constants $C_1(d; \alpha, \beta, \lambda)$, $C_2(d; \alpha, \beta, \lambda)$ for which the minimal value of a constant $\leq V_{\alpha, \beta, \lambda}$ from the inequality

$$\|I_{\alpha, \beta, \lambda}[f]\| \|G(\psi_{\alpha, \beta, \lambda})\| \leq V_{\alpha, \beta, \lambda} \|f\| \|G(\psi)\|$$

lies in the closed interval

$$V_{\alpha, \beta, \lambda} \in [C_1(d; \alpha, \beta, \lambda), C_2(d; \alpha, \beta, \lambda)]. \quad (25)$$

Proof follows immediately from the main results of papers [35], [36]. There are described, for instance, correspondent examples.

Note that in the case $\alpha = \beta = 0$ we can conclude that

$$\frac{C_3(d)}{d - \lambda} \leq V_{0,0,\lambda} \leq \frac{C_4(d)}{d - \lambda},$$

see [35].

□

5. SINGULAR OPERATORS OF HARDY TYPE AND ITS WEIGHT GENERALIZATIONS

We consider in this section the operators of a view

$$U_\lambda[f](t) = t^{\lambda-1} \int_0^t s^{-\lambda} f(s) ds.$$

Here $X = R_+$, $\lambda = \text{const} \in (-\infty, 1)$, and we equip all the Borelian sets of the space X by the measure

$$\nu(A) = \int_A \frac{dt}{t}.$$

This operators play a very important role in the theory of operators interpolation [3], chapter 3, section 1.

There it is proved the following $L_{p,\nu}$ estimation:

$$\|U_\lambda[f]\|_{p,\nu} \leq \frac{1}{1-\lambda} \|f\|_{p,\nu}, \quad p \in [1, \infty].$$

Therefore, if we consider arbitrary function $\psi \in G\Psi(1, \infty)$, we conclude on the basis of theorem 1 that

$$\|U_\lambda[f]\|G(\psi) \leq \frac{1}{1-\lambda} \|f\|G(\psi). \quad (26)$$

Theorem 7. The constant $1/(1-\lambda)$ in the inequality (26) is exact.

Proof is at the same as before. Namely, let us consider the function

$$f_0(t) = |\log t|^{-\Delta} I(t \in (0, 1)), \quad \Delta = \text{const} \in (0, 1);$$

then

$$|f_0|_{p,\nu} = (p\Delta - 1)^{-1/p}, \quad p \in (1/\Delta, \infty),$$

and we get as $t \rightarrow 0+$:

$$U_\lambda[f_0](t) = t^{\lambda-1} \int_0^t s^{-\lambda} |\log s|^{-\Delta} ds \sim$$

$$\frac{1}{1-\lambda} |\log t|^{-\Delta} = \frac{1}{1-\lambda} f(t).$$

Therefore, we have as $p \rightarrow 1/\Delta + 0$

$$|U_\lambda[f_0]|_p \sim \frac{1}{1-\lambda} |f_0|_p.$$

This completes the proof of this theorem.

Another version of Hardy's inequality see in [40], pp. 134-135:

$$\begin{aligned} & \left(\int_0^\infty t^{-p/p_1} \left(\int_0^t |f(s)| s^{1/p_1-1} ds \right) dt \right)^{1/p} \leq \\ & (1/p_1 - 1/p)^{-1} \left(\int_0^\infty |f(s)|^p ds \right)^{1/p}; \end{aligned} \quad (27a)$$

$$\begin{aligned} & \left(\int_0^\infty t^{-p/p_0} \left(\int_t^\infty |f(s)| s^{1/p_0-1} ds \right) dt \right)^{1/p} \leq \\ & (1/p - 1/p_0)^{-1} \left(\int_0^\infty |f(s)|^p ds \right)^{1/p}; \end{aligned} \quad (27b)$$

here $1 \leq p_1 < p < p_0 < \infty$.

The inequalities (27a) and (27b) may be rewritten as follows.

$$U_\alpha[f]_p \leq (\alpha - 1/p)^{-\alpha} |f|_p, \quad p \in (1/\alpha, \infty),$$

$$U_\alpha[f]_p \leq (1/p - \alpha)^{-\alpha} |f|_p, \quad p \in (1, 1/\alpha),$$

but here $\alpha \in (0, 1)$.

If the function $\psi = \psi(p)$ belongs to the set $G\Psi(1/\alpha, \infty)$, we introduce the auxiliary function

$$\psi_\alpha(p) = (\alpha - 1/p)^{-\alpha} \psi(p).$$

It follows from the theorem 1 that

$$\|U_\alpha[f]\|G(\psi_\alpha) \leq 1 \cdot \|f\|G(\psi). \quad (28)$$

Theorem 8. The constant 1 in the inequality (28) is exact.

Proof is at the same as in the theorem 8. Namely, we consider a function

$$f_\Delta(x) = x^{-\alpha} (\log x)^\Delta I(x > 1), \quad \Delta = \text{const} \geq 1;$$

then we have as $p \rightarrow 1/\alpha + 0$ and $x \rightarrow \infty$, $x \geq 1$:

$$|f_0|_p = (\alpha p - 1)^{-\Delta-1/p} \Gamma^{1/p}(\Delta p + 1) \sim (\alpha p - 1)^{-\Delta-\alpha} \Gamma^\alpha(\Delta/\alpha + 1);$$

$$u_\Delta(x) := x^{-\alpha} \int_0^x s^{\alpha-1} s^{-\alpha} (\log s)^\Delta ds =$$

$$(\Delta + 1)^{-1} x^{-\alpha} (\log x)^{\Delta+1},$$

$$|u_\Delta(\cdot)|_p \sim \frac{1}{\Delta + 1} \frac{\Gamma^\alpha((\Delta + 1)/\alpha + 1)}{(\alpha p - 1)^{(\Delta+1)/\alpha}}.$$

Note that as $p \rightarrow 1/\alpha + 0$

$$\overline{\lim}_{p \rightarrow 1/\alpha + 0} \frac{|u_\Delta|_p}{|f_\Delta|_p} : (\alpha - 1/p) \geq e^{-1} \left(\frac{\Delta + 1}{\Delta} \right)^\Delta. \quad (29)$$

The expression (29) tends to one as $\Delta \rightarrow \infty$.

This completes the proof of this theorem.

The case when $p \in (1, 1/\alpha)$ may be considered analogously, by mean of an example

$$g_\Delta(x) = x^{-\alpha} |\log x|^\Delta I(x \in (0, 1)).$$

□

6. SINGULAR OPERATORS OVER THE SPACES WITH WEIGHT

Let r be arbitrary constant positive number, $r \neq 1$. Let us consider in this section integral operators of a view

$$f_{(1)}[f](x) = \int_0^x f(t) dt, \quad X = R_+, \quad r > 1,$$

or

$$f^{(1)}[f](x) = \int_x^\infty f(t) dt, \quad X = R_+, \quad r < 1.$$

i.e. we consider the integral of $f(\cdot)$.

We equip the Borelian sets of the space $X = R_+$ by means of the weight measure $\mu_r(A)$:

$$\mu_r(A) = \int_A x^{-r} dx$$

and will denote for simplicity the L_p norm of the function $h : R_+ \rightarrow R$ in this space as $|h|_{p,r}$:

$$|h|_{p,r} := \left[\int_0^\infty x^{-r} |h(x)|^p dx \right]^{1/p}.$$

In the book [19], p. 227 - 231 it is proved:

$$\int_0^\infty x^{-r} |f^{(1)}(x)|^p dx \leq \left(\frac{p}{|r-1|} \right)^p \cdot \int_0^\infty x^{-r} (xf(x))^p dx. \quad (30)$$

The inequality (30) may be rewritten as follows:

$$|F|_{p,r} \leq \frac{p}{|r-1|} \cdot |g|_{p,r},$$

where

$$F[g](x) = \int_0^x y^{-1} g(y) dy, \quad r > 1;$$

the case when $r < 1$ and

$$F[g](x) = \int_x^\infty y^{-1} g(y) dy$$

is considered analogously and will be omitted.

Therefore, in the considered case $r < 1$.

Let $\psi(\cdot)$ be any function from the set $G\Psi(1, \infty)$; we define

$$\psi_1(p) = p \psi(p).$$

Theorem 9.

$$\|F[g]\|G(\psi_1) \leq (|r-1|)^{-1} \|g\|G(\psi),$$

where the value of the constant $(|r-1|)^{-1}$ is the best possible.

Proof of the low estimate is ordinary. Let us consider the example of a function:

$$g_\Delta(x) = |\log x|^\Delta I(x \in (0, 1)), \quad \Delta = \text{const} > 1;$$

then

$$|g_\Delta|_{p,r}^p = \int_0^1 x^{-r} |\log x|^{\Delta p} dx = \frac{\Gamma(\Delta p + 1)}{|1-r|^{p\Delta+1}},$$

and for the values x from the interval $x \in (0, 1)$:

$$F[g_\Delta](x) = \int_0^x y^{-1} |\log y|^\Delta dy = (\Delta + 1)^{-1} |\log x|^{\Delta+1};$$

$$|F[g_\Delta]|_{p,r} \geq (\Delta + 1)^{-1} \frac{\Gamma^{1/p}((\Delta + 1)p + 1)}{|1-r|^{\Delta+1+1/p}}.$$

It is easy to calculate using Stirling's formula that as $p \rightarrow \infty$

$$\overline{\lim} [|r-1| |F[g_\Delta]|_p / |g_\Delta|_p] \geq e^{-1} (1 + 1/\Delta)^\Delta.$$

The assertion of our theorem follows from the equality

$$\lim_{\Delta \rightarrow \infty} e^{-1} (1 + 1/\Delta)^\Delta = 1.$$

□

7. FOURIER INTEGRAL OPERATORS

We consider here the classical Fourier integral operator of a view

$$F[f](x) = F(x) = \int_{-\infty}^{\infty} \exp(itx) f(t) dt.$$

It is known [44], pp. 96-98, that

$$\int_R |F(x)|^p |x|^{p-2} dx \leq p^2/(p-1) \int_R |f(t)|^p dt, \quad p \in (1, \infty). \quad (31)$$

The inequality (31) may be rewritten in the language of BGLS as follows. Let ψ be arbitrary function from the class $G\Psi(1, \infty)$. We define the new measure $\mu(dx) = dx/x^2$ and introduce the new function

$$\psi_F(p) = p^2 \psi(p)/(p-1),$$

then it follows from the theorem 1 that

$$||F[f]||G(\psi_F, \mu) \leq 1 \cdot ||f||G(\psi). \quad (32)$$

Theorem 10. The constant 1 in the inequality (32) is the best possible.

Proof. Let us consider the example function

$$f_0(x) = x^{-1} I(x > 1),$$

then

$$|f_0|_p = (p-1)^{1/p}.$$

Further, we have as $t \rightarrow 0$, $t \in (0, 1)$:

$$\begin{aligned} F[f_0](t) &= \int_1^{\infty} \exp(itx) dx/x = \int_t^{\infty} \exp(iy) dy/y = C + \\ &\int_t^1 \exp(iy) dy/y \sim |\log t|. \end{aligned}$$

Therefore, as $p \rightarrow 1 + 0$

$$|F[f_0]|_{p,\mu}^p \sim \int_0^1 |\log t|^p t^{p-2} dt = \int_0^{\infty} e^{-y(p-1)} y^p dy = (p-1)^{-p-1} \Gamma(p+1);$$

$$|F[f_0]|_{p,\mu} \sim (p-1)^{-1-1/p}$$

and we have as before choosing $\psi(p) = |f_0|_p$

$$\sup_{p \in (1, \infty)} \frac{|F[f_0]|_{p,\mu}}{|f_0|_p} \times \frac{p-1}{p^2} \geq \overline{\lim}_{p \rightarrow 1} \frac{|F[f_0]|_{p,\mu}}{|f_0|_p} \times \frac{p-1}{p^2} = 1.$$

□

8. RIESZ SINGULAR INTEGRAL OPERATOR

The operator of a view

$$R_j[f](x) = c(n) \text{ p.v. } \int_{R^n} \frac{(x_j - y_j)f(y) dy}{|x - y|^{n+1}}$$

is named Riesz transform, or Riesz operator. Here $n = 1, 2, \dots; j = 1, 2, \dots, n$;

$$c(n) = \pi^{-(n+1)/2} \Gamma((n+1)/2).$$

Note that the general case of singular integral operators acting in the BGLS spaces without exact constant calculation is considered in [34].

In the one-dimensional case, i.e. when $n = 1$ the Riesz transform coincides with the so-called Hilbert transform

$$Rf(x) = Hf(x) \stackrel{\text{def}}{=} \pi^{-1} \text{ p.v. } \int_{-\infty}^{\infty} \frac{f(y) dy}{x - y}.$$

The exact value of the $L_p \rightarrow L_p$ norm of the Riesz operator does not depend on the dimension n . Namely, we have for the values p from the open interval $p \in (1, \infty)$:

$$h(p) \stackrel{\text{def}}{=} \|R_j\|(L_p \rightarrow L_p) = \tan(\pi/(2p)), \quad p \in (1, 2]$$

and

$$h(p) = 1/\tan(\pi/(2p)), \quad p \in [2, \infty),$$

see [38], [22], chapter 12, section 12.1.

Note that as $p \rightarrow 1 + 0$ $h(p) \sim 1/(p-1)$ and as $p \rightarrow \infty$ $h(p) \sim p$.

At the same estimation for the $L_p \rightarrow L_p$ norm is true for the Hilbert transform in the case $X = (-\pi, \pi)$, where the operator H is defined as follows:

$$H_\pi[f](x) = (2\pi)^{-1} \text{ p.v. } \int_{-\pi}^{\pi} \frac{f(x-y)}{\tan(y/2)} dy, \quad x-y = x-y \pmod{2\pi}.$$

Note also that if

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)),$$

then

$$H_\pi[f](x) = \sum_{k=1}^{\infty} (a_k \sin(kx) - b_k \cos(kx)).$$

Let us denote for arbitrary function $\psi \in G\Psi(1, \infty)$ the new function from at the same set

$$\psi_H(p) = h(p) \cdot \psi(p).$$

Theorem 11

$$\|R_j f\|_{G(\psi_H)} \leq Z \cdot \|f\|_{G(\psi)},$$

where in general case the exact value of the constant Z is equal to $2/\pi$:

$$\overline{Z} \stackrel{def}{=} \sup_{\psi \in G\Psi(1, \infty)} \sup_{f \in G\psi(1, \infty), f \neq 0} \frac{\|R_j f\| G(\psi_H)}{\|f\| G(\psi)} = \frac{2}{\pi}.$$

Proof. The upper bound for the constant Z it follows from the exact value for the function $h(\cdot)$; it remain to provide the low bound.

It is sufficient to consider only the one-dimensional case, i.e. when the Riesz transform coincides with the Hilbert transform.

A. We consider at first the case $X = (-\pi, \pi)$. Let us consider the *family* of a functions of a view:

$$g_\Delta(x) = \sum_{n=2}^{\infty} n^{-1} (\log n)^\Delta \sin(nx).$$

Here $\Delta = \text{const}$ is some "great" constant.

It is proved in [46], p. 182-185 that as $x \rightarrow 0$

$$|g_\Delta(x)| \sim 0.5\pi |\log |x||^\Delta,$$

therefore

$$|(2/\pi) g_\Delta|_p^p \asymp 2 \int_0^1 |\log x|^p dx = 2\Gamma(\Delta p + 1);$$

and as $p \rightarrow \infty$

$$|(2/\pi) g_\Delta|_p \sim \Gamma^{1/p}(\Delta p + 1).$$

The correspondent Hilbert transform for the function g_Δ , which we denote by $-u_\Delta$, has a view

$$u_\Delta(x) = \sum_{n=2}^{\infty} n^{-1} (\log n)^\Delta \cos(nx).$$

It is proved also in the book of Zygmund [46], p. 182-185 that as $x \rightarrow 0$

$$|u_\Delta(x)| \sim (\Delta + 1)^{-1} |\log |x||^{\Delta+1},$$

and following as $p \rightarrow \infty$

$$|u_\Delta|_p \sim (\Delta + 1)^{-1} \Gamma^{1/p}((\Delta + 1)p + 1).$$

Let us denote

$$V^{(\infty)}(g, p) = \frac{|T_H[g]|_p}{|h(p) g|_p}, \quad p \in (1, \infty),$$

$$\overline{V^{(\infty)}} = \sup_{g \in L(1, \infty)} \sup_{p \in (1, \infty)} V^{(\infty)}(g, p).$$

It follows from the upper bounds of this theorem that $\overline{V^{(\infty)}} \leq 2/\pi$. On the other hand, we have for all the values $\Delta = \text{const} > 0$ using the famous Stirling's formula:

$$\begin{aligned} \overline{V^{(\infty)}} &\geq \overline{\lim}_{\Delta \rightarrow \infty} \overline{\lim}_{p \rightarrow \infty} \frac{|T_H[g_\Delta]|_p}{|h(p) g_\Delta|_p} \geq \\ &\frac{2}{\pi} \overline{\lim}_{\Delta \rightarrow \infty} \overline{\lim}_{p \rightarrow \infty} \frac{\Gamma^{1/p}((\Delta + 1)p + 1)}{p \cdot \Gamma^{1/p}(\Delta p + 1)} \geq \end{aligned}$$

$$\overline{\lim}_{\Delta \rightarrow \infty} \frac{2}{\pi} e^{-1} \left(\frac{\Delta + 1}{\Delta} \right)^\Delta = \lim_{\Delta \rightarrow \infty} \frac{2}{\pi} e^{-1} \left(\frac{\Delta + 1}{\Delta} \right)^\Delta = \frac{2}{\pi}.$$

Choosing as a function $\psi(p)$ the following expression:

$$\psi(p) := \psi_0(p) \stackrel{def}{=} |g_\Delta|_p, \quad p \in (1, \infty),$$

for sufficiently great values Δ , we complete the consideration of the case **A** of this theorem.

B. In this pilcrow we consider the case of the Hilbert's transform on the whole real axis. Let us choose the function

$$f_1(x) = I(x \in (c, d)), \quad x \in R, \quad c, d = \text{const}, \quad d = c + 1;$$

then $\forall p \geq 1 \quad |f_1|_p = 1$.

The correspondent Hilbert transform for the function $f_1(x)$ we denote by $v_1(x)$; it is equal to

$$v_1(x) = \frac{1}{\pi} \log \left| \frac{x - c}{x - d} \right|,$$

see [3], p. 143-144.

It is easy to see that as $|x| \rightarrow \infty$

$$v_1(x) \sim \frac{1}{\pi} |x|^{-1},$$

and hence

$$|v_1|_p \sim \frac{2}{\pi} \frac{1}{p-1}, \quad p \rightarrow 1+0.$$

Let us denote as before

$$V_1(f, p) = \frac{|T_H[f]|_p}{|h(p) f|_p}, \quad p \in (1, \infty),$$

$$\overline{V}_1 = \sup_{f \in L(1, \infty)} \sup_{p \in (1, \infty)} V_1(f, p).$$

It follows from the upper bounds of this theorem that $\overline{V}_1 \leq 2/\pi$, but

$$\overline{V}_1 \geq \overline{\lim}_{p \rightarrow 1+0} V_1(f_1, p) = \frac{2}{\pi}.$$

Thus, $\overline{V}_1 = 2/\pi$.

This completes the proof of our theorem.

□

9. DISCRETE CASE

We consider in this section the case when $X = \{1, 2, 3, \dots\}$ and μ is ordinary counting measure. So, the L_p norm of the function $f = f(k)$, $k \in X$ may be defined as follows:

$$|f|_p = \left(\sum_{k=1}^{\infty} |f(k)|^p \right)^{1/p}, \quad p \geq 1.$$

In the terminology of a book [3] these spaces are called *discrete resonant measurable spaces*.

First of all we will formulate some simple properties of the L_p norms in this space X .

1. $p \leq q \Rightarrow |f|_q \leq |f|_p$;
2. If for some $p < \infty$ $|f|_p < \infty$, then

$$\lim_{p \rightarrow \infty} |f|_p = \sup_{k \geq 1} |f(k)| \stackrel{\text{def}}{=} |f|_{\infty};$$

3. Natural functions.

Let $f \in l_s$ for some value s , $s \in [1, \infty)$; then $\forall q \geq p$ $|f|_q \leq |f|_s$. Therefore, if we define the natural function for the vector f , i.e. the function

$$\psi_f(p) = |f|_p, \quad s < p \leq \infty,$$

then the function $\psi_f(p)$ is bounded in the set $p \in (s_1, \infty)$ and moreover

$$\exists \lim_{p \rightarrow \infty} \psi_f(p) = \sup_k |f(k)| < \infty.$$

We will consider in this section only the functions with the last both Properties.

Example. Let $f(k) = k^{-\alpha}$, $\alpha \in (0, 1)$. We get:

$$\psi_f(p) \asymp \left[\frac{p}{p - 1/\alpha} \right]^{1/\alpha}, \quad p \in (1/\alpha, \infty).$$

4. Tail behavior; see for comparison [34].

Let us denote for the (infinite) sequence $f = \{f(k)\}$ the so-called tail function

$$Z_f(\epsilon) = \mu\{k : |f(k)| > \epsilon\} = \text{card}\{k : |f(k)| > \epsilon\}, \quad \epsilon \geq 0.$$

Obviously, $Z_f(\epsilon) = 0$, if $\epsilon > \sup_k |f(k)|$.

It follows from the Tchebyshevs inequality that if the sequence $f = \{f(k)\}$ belong to the space $G(\psi)$, then

$$Z_f(\epsilon) \leq \inf_p \left[\frac{|f|_p^p \cdot \psi(p)}{\epsilon^p} \right].$$

Inversely,

$$|f|_p = \left[p \int_0^{\infty} y^{p-1} T(y) dy \right]^{1/p},$$

Therefore

$$\|f\|G(\psi) = \sup_{p \in (a,b)} \left\{ \left[p \int_0^\infty y^{p-1} T(y) dy \right]^{1/p} / \psi(p) \right\}.$$

Example. If $f(k) = k^{-\alpha}$, $k = 1, 2, \dots$, $\alpha \in (0, 1)$; then

$$\psi_f(p) \asymp [p/(p-1/\alpha)]^{1/\alpha};$$

$$Z_f(\epsilon) \asymp \epsilon^{-1/\alpha}, \quad \epsilon \rightarrow 0+.$$

But if it is given that

$$|f|_p \leq \left[\frac{p}{p-1/\alpha} \right]^{1/\alpha},$$

then we can conclude only

$$Z_f(\epsilon) \asymp C \cdot \left[\frac{\epsilon}{|\log \epsilon|} \right]^{-1/\alpha}, \quad \epsilon \in (0, 1/e).$$

In the classical book [19] there are many examples of $l_p \rightarrow l_p$ norm estimations for matrix linear operators, which are completely analogous to the "continuous" case, as in section 3. For example:

$$T_0^{(d)}[f](n) = n^{-1} \sum_{k=1}^n f(k), \quad |T_0^{(d)}|_{p,p} = p/(p-1);$$

$$T_+^{(d)}[f](n) := \sum_{k=1}^\infty \frac{f(k)}{k+n}, \quad |T_+^{(d)}|_{p,p} = \frac{\pi}{\sin(\pi/p)}, \quad p \in (1, \infty);$$

$$T_m^{(d)}[f](n) := \sum_{k=1}^\infty \frac{f(k)}{\max(k, n)}, \quad |T_m^{(d)}|_{p,p} = \frac{p^2}{p-1}, \quad p \in (1, \infty),$$

$$T_d^{(d)}[f](n) := \sum_{k=n}^\infty \frac{f(k)}{k}, \quad |T_d^{(d)}|_{p,p} = p, \quad p \in (1, \infty),$$

$$T_l^{(d)}[f](n) = \sum_{k=1}^\infty \frac{\log(k/n)f(k)}{k-n}, \quad 0/0 = 0, \quad |T_l^{(d)}|(p, p) = \left[\frac{\pi}{\sin(\pi/p)} \right]^2,$$

etc.

We will consider more generally linear matrix operators of a view:

$$T_H^{(d)}[f](n) = n^{-1} \sum_{k=1}^\infty f(k)H(k/n),$$

where the function $H = H(z)$, $z \in (0, \infty)$ satisfies the conditions of lemmas 1a or 1b or 2a or 2b. Suppose in addition the function $H(\cdot)$ is piecewise strictly monotonically decreasing and piecewise continuous with finite points of charging.

Theorem 12.

$$\|T_H^{(d)} f\|G(\psi_{(\phi)}) \leq V_H \|f\|G(\psi),$$

where the exact value of constant V_H is equal to one.

Proof is at the same as in the proof of theorems 3 and 4, with at the same "counter-examples"; more exactly, if the function $f = f(x), x \in (0, \infty)$ is some example in the theorems 3 and 4, then the sequence

$$f_0(k) = f(k)$$

is the correspondent "counter-example" in the discrete case.

Many another generalizations see in the works [4], [5], [6], [7], [17], [18]; see also reference therein.

We will consider the linear operator with lower triangular (infinite) matrix $A = \{a(n, k)\}$ with non-negative entries: $a(n, k) \geq 0$ of a view:

$$a(n, k) = \lambda(k)/\Lambda(n), 1 \leq k \leq n; a(n, k) = 0, k \geq n + 1;$$

here

$$\lambda(k) \geq 0, \lambda(1) > 0, \Lambda(n) = \sum_{k=1}^n \lambda(k).$$

The correspondent linear operator $T = T_A$ may be defined as ordinary, indeed, for the infinite sequence $x = \{x(i), i = 1, 2, 3, \dots\}$

$$T_A[x](j) = \sum_{i=1}^j a(j, i)x(i).$$

It is proved in the article [4], see also [17], [18] that if

$$L := \sup_n \left(\frac{\Lambda(n+1)}{\lambda(n+1)} - \frac{\Lambda(n)}{\lambda(n)} \right) \in (1, \infty),$$

then for the values $p, p > L$

$$|T_A|_{p,p} \leq \frac{p}{p-L}.$$

As a consequence, if the function $\psi(p)$ belongs to the set $G\Psi(L, \infty)$ and if we define a new function

$$\psi^{(L)}(p) = \frac{p}{p-L} \psi(p),$$

then it follows from theorem 1 that for arbitrary sequence $x = \{x(i), i = 1, 2, \dots\}$ belonging to the space $G(\psi)$

$$||T_A[x]||G(\psi^{(L)}) \leq 1 \cdot ||x||G(\psi). \quad (33)$$

Theorem 13. The constant 1 in the inequality (33) is in general case exact.

Analogously may be considered the case when $\alpha = \text{const} \in (0, 1)$ and

$$a(n, k) = n^{-\alpha} [k^\alpha - (k-1)^\alpha]$$

or

$$a(n, k) = \frac{k^{\alpha-1}}{\sum_{i=1}^n i^{\alpha-1}}.$$

In both the last cases the $l_p \rightarrow l_p$ norm of the operator with the matrix $A = \{a(n, k)\}$ allows the following estimation:

$$|T_A|_{p,p} \leq \frac{\alpha p}{\alpha p - 1}, \quad p > 1/\alpha.$$

As before, if the function $\psi(p)$ belongs to the set $G\Psi(1/\alpha, \infty)$ and if we define a new function

$$\psi_{(\alpha)}(p) = \frac{\alpha p}{\alpha p - 1} \psi(p),$$

then it follows from theorem 1 that for arbitrary sequence $x = \{x(i), i = 1, 2, \dots\}$ belonging to the space $G(\psi)$

$$||T_A[x]||G(\psi_{(\alpha)}) \leq 1 \cdot ||x||G(\psi). \quad (34)$$

Theorem 14. The constant 1 in the inequality (34) is exact.

Proof of theorems 13 and 14. It is sufficient to prove the last theorem. Let us consider as ordinary the following example:

$$x = \{x(k)\}, \quad x(k) = k^{-\alpha}(\log k)^\Delta, \quad k = 1, 2, \dots; \quad \Delta = \text{const} > 0.$$

We get for the values $p \rightarrow 1/\alpha + 0$:

$$|x|_p^p \sim (\alpha p - 1)^{-1-\Delta p} \Gamma(\Delta p + 1),$$

$$|x|_p \sim (\alpha p - 1)^{-\Delta-1/p} \Gamma^{1/p}(\Delta p + 1) \sim (\alpha p - 1)^{-\Delta-\alpha} \Gamma^\alpha(\Delta/\alpha + 1);$$

$$y := T_A[x] = \{y(n)\}, \quad n \rightarrow \infty \Rightarrow y(n) \sim (\Delta + 1)^{-1} \alpha n^{-\alpha} (\log(n))^{\Delta+1};$$

$$|y|_p \sim (\Delta + 1)^{-1} \alpha (\alpha p - 1)^{-\Delta-\alpha-1} \Gamma^\alpha((\Delta + 1)/\alpha + 1);$$

$$\begin{aligned} \overline{\lim}_{p \rightarrow 1/\alpha+0} \left[\frac{|y|_p}{|x|_p} : \frac{\alpha p}{\alpha p - 1} \right] &\geq e^{-1} \left(1 + \frac{1}{\Delta} \right)^\Delta; \\ \lim_{\Delta \rightarrow \infty} e^{-1} \left(1 + \frac{1}{\Delta} \right)^\Delta &= 1. \end{aligned}$$

□

10. CONCLUDING REMARKS

The assertion of theorem 1 is true still for the so-called sublinear operators, i.e. for the operators $T[f]$ with properties:

$$|T[f + g]|_p \leq |T[f]|_p + |T[g]|_p, \quad p \in (1, \infty);$$

$$|T[\lambda f]|_p = |\lambda| |T[f]|_p, \quad \lambda = \text{const}.$$

Many examples of sublinear operators give us maximal operators. Let $T_\theta[\cdot]$, $\theta \in \Theta$ be a family of linear operators. We put:

$$T[f](x) = \sup_{\theta \in \Theta} |T_\theta[f](x)|,$$

if there exists and satisfies, e.g. the inequality

$$|T[f]|_p \leq \Phi(p) |f|_p,$$

where $0 < \Phi(p) < \infty$, $p \in (1, \infty)$.

For instance, let $X = (-\pi, \pi)$ and let for any function $f : X \rightarrow R$ $S^*[f]$ be the maximum of the absolute value of the partial Fourier $S_n[f]$ sum for the function f :

$$S^*[f] = \sup_n |S_n[f](x)|.$$

It is known, see for example, [40], p. 151, that for some absolute constant $C_1 \in (0, \infty)$

$$|S^*[f]|_p \leq C_1 \frac{p^4}{(p-1)^3} |f|_p, \quad p \in (1, \infty),$$

i.e. in this case $\Phi(p) = p^4/(p-1)^3$.

Another example. Let $H^*[f]$ be the maximal Hilbert transform for the function f , $f : R \rightarrow R$. It is proved in [40], p. 84 that for some absolute constant $C_2 \in (0, \infty)$

$$|H^*[f]|_p \leq C_2 \frac{p^2}{p-1} |f|_p,$$

i.e. here $\Phi(p) = p^2/(p-1)$.

It follows from theorem 1 that if in the case when $\Phi(p) < \infty$, $p \in (1, \infty)$ and $\psi(\cdot) \in G\Psi(1, \infty)$ we define

$$\tilde{\psi}(p) = \Phi(p) \cdot \psi(p),$$

then

$$||T[f]||G(\tilde{\psi}) \leq ||f||G(\psi).$$

Analogous asymptotically as $p \rightarrow 1 + 0$ and $p \rightarrow \infty$ exact estimations for the function $\Phi = \Phi(p)$ are obtained for the famous singular operators of Calderon - Zygmund type or analogously maximal Calderon - Zygmund type, see [1], [3], [11], [42] etc.

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